

STABLE COHOMOLOGY OF ALTERNATING GROUPS

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ABSTRACT. In this article we determine the stable cohomology groups $H_s^i(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z})$ of the alternating groups \mathfrak{A}_n for all integers n and i , and all primes p .

1. INTRODUCTION AND PRELIMINARIES

Let G be a finite group, V a finite-dimensional generically free complex representation of G , and let $V^L \subset V$ be the nonempty open subset of V on which the G -action is free. There is then a natural homotopy class of maps from the classifying space BG to V^L/G which, for each nonempty G -invariant Zariski open subset $U \subset V^L$ gives maps $H^i(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^i(U/G, \mathbb{Z}/p\mathbb{Z})$. It turns out that the kernel $K_{G,V}$ of

$$H^i(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow \varinjlim_U H^i(U/G, \mathbb{Z}/p\mathbb{Z})$$

is independent of V , and the stable cohomology $H_s^i(G, \mathbb{Z}/p\mathbb{Z})$ is defined to be the quotient $H^i(G, \mathbb{Z}/p\mathbb{Z})/K_{G,V}$. Let $K = \mathbb{C}(V)^G$. Algebraically, $H_s^i(G, \mathbb{Z}/p\mathbb{Z})$ can be identified with the image of $H^i(G, \mathbb{Z}/p\mathbb{Z})$ in $H^i(\text{Gal}(K), \mathbb{Z}/p\mathbb{Z})$.

More accessible computationally and stable birational invariants of the function field K are the unramified cohomology groups $H_{\text{nr}}^i(G, \mathbb{Z}/p\mathbb{Z})$ defined as follows: geometrically, the unramified cohomology classes a inside $H_s^i(G, \mathbb{Z}/p\mathbb{Z})$ are those for which, given any divisorial valuation ν_D of K , there exists a normal model $X = X_D$ of K on which ν_D has a center, an isomorphism $i : U_X \rightarrow U_{V^L/G}$ between nonempty open subsets U_X of X and $U_{V^L/G}$ of V^L/G , and a representative a' of a in $H^i(U_{V^L/G}, \mathbb{Z}/p\mathbb{Z})$ such that there is a class $b \in H^i(X, \mathbb{Z}/p\mathbb{Z})$ whose image in $H^i(U_X, \mathbb{Z}/p\mathbb{Z})$ coincides with $i^*(a')$. More algebraically, if $\mathcal{O}_\nu \subset K$ is the valuation ring of ν , $\kappa_\nu = \mathcal{O}_\nu/\mathfrak{m}_\nu$ its residue field, $S = \text{Spec}(\mathcal{O}_\nu)$ with open subset the generic point $U = \text{Spec}(K) \subset S$

¹ Supported by NSF grant DMS-1001662 and by AG Laboratory GU- HSE grant RF government ag. 11 11.G34.31.0023.

² Supported by the German Research Foundation (Deutsche Forschungsgemeinschaft) through Heisenberg-Stipendium BO 3699/1-1.

and complement the closed point $Z = \operatorname{Spec}(\kappa_\nu) \subset S$, one can write down the long exact sequence of étale cohomology with supports

$$\cdots \rightarrow H^i(S, \mathbb{Z}/p) \rightarrow H^i(U, \mathbb{Z}/p) \rightarrow H_Z^{i+1}(S, \mathbb{Z}/p) \rightarrow H^{i+1}(S, \mathbb{Z}/p) \rightarrow \cdots$$

where $H^i(U, \mathbb{Z}/p) \simeq H^i(\operatorname{Gal}(K), \mathbb{Z}/p)$ and there is the cohomological purity isomorphism

$$H_Z^j(S, \mathbb{Z}/p) \simeq H^{j-2}(Z, \mathbb{Z}/p)$$

whence the preceding sequence becomes the *Gysin sequence*

$$\begin{aligned} \cdots \rightarrow H_{\text{ét}}^i(S, \mathbb{Z}/p) &\xrightarrow{r_\nu} H^i(\operatorname{Gal}(K), \mathbb{Z}/p) \\ &\xrightarrow{\partial_\nu} H^{i-1}(\operatorname{Gal}(\kappa_\nu), \mathbb{Z}/p) \rightarrow H_{\text{ét}}^{i+1}(S, \mathbb{Z}/p) \rightarrow \cdots \end{aligned}$$

A class in $H^i(\operatorname{Gal}(K), \mathbb{Z}/p)$ is clearly unramified according to the geometric definition if and only if it is in the image of all maps r_ν for ν running over the divisorial valuations of K , i.e. equivalently if it is in the kernel of all maps ∂_ν , the *residue maps*. The preceding sequence has as topological analogue the Borel-Moore long exact sequence. The residue map

$$\partial_\nu : H^i(\operatorname{Gal}(K), \mathbb{Z}/p) \rightarrow H^{i-1}(\operatorname{Gal}(\kappa_\nu), \mathbb{Z}/p)$$

agrees -up to a sign- with the following map defined entirely within the framework of Galois cohomology (see e.g. [GMS], Chap. II of Serre's part, §6 and §7): extend ν in some way to a valuation ν^* on \bar{K} which is possible by Chevalley's theorem; all such extensions are conjugate under $\Gamma = \operatorname{Gal}(K)$, and ν^* defines subgroups $\Gamma_Z \subset \Gamma$ (the decomposition group, *Zerlegungsgruppe*) and $\Gamma_T \subset \Gamma$ (the inertia subgroup, *Trägheitsgruppe*) by the conditions that $\sigma \in \Gamma$ is in Γ_Z if $\sigma \cdot \nu^*$ and ν^* are equivalent valuations, i.e. have the same valuation ring, and Γ_T consists of those σ such that $\sigma \cdot x - x \in \mathfrak{M}_{\nu^*}$ for all x in the valuation ring of ν^* whose maximal ideal we denoted by \mathfrak{M}_{ν^*} . The decomposition group can be identified with the Galois group $\operatorname{Gal}(\bar{K}_\nu/K_\nu)$ of the completion of K with respect to ν . The residue map ∂_ν then factors over the restriction to the decomposition group

$$\partial_\nu : H^i(\operatorname{Gal}(K), \mathbb{Z}/p) \rightarrow H^i(\operatorname{Gal}(K_\nu), \mathbb{Z}/p) \xrightarrow{r} H^{i-1}(\operatorname{Gal}(\kappa_\nu), \mathbb{Z}/p)$$

where the second arrow has the following description in the local situation: the Galois group $\Gamma_{K_\nu} = \operatorname{Gal}(K_\nu)$ sits in the exact sequence

$$1 \rightarrow I \rightarrow \Gamma_{K_\nu} \rightarrow \Gamma_{\kappa_\nu} = \operatorname{Gal}(\kappa_\nu) \rightarrow 1$$

where the surjection is given by the fact that ν extends uniquely to \bar{K}_ν and the residue field of the extension is an algebraic closure of κ_ν . The kernel is the inertia subgroup which we denote by I in this

context, and it is topologically cyclic, $I \simeq \hat{\mathbb{Z}}$, corresponding to taking roots of the uniformizing parameter, and the preceding sequence splits, $\Gamma_{K_\nu} \simeq \hat{\mathbb{Z}} \oplus \text{Gal}(\kappa_\nu)$. As $\hat{\mathbb{Z}}$ has cohomological dimension 1, one gets $H^i(\Gamma_{K_\nu}, \mathbb{Z}/p) \simeq H^i(\Gamma_{\kappa_\nu}, \mathbb{Z}/p\mathbb{Z}) \oplus H^{i-1}(\Gamma_{\kappa_\nu}, \mathbb{Z}/p\mathbb{Z})$ and a projection, which is independent of the splitting, $H^i(\text{Gal}(K_\nu), \mathbb{Z}/p) \xrightarrow{r} H^{i-1}(\text{Gal}(\kappa_\nu), \mathbb{Z}/p)$, defining the second arrow in the sequence of maps yielding ∂_ν . More precisely, the Hochschild-Serre spectral sequence of the group extension of Γ_{κ_ν} by I

$$H^p(\Gamma_{\kappa_\nu}, H^q(I, \mathbb{Z}/p)) \implies H(\Gamma_{K_\nu}, \mathbb{Z}/p)$$

reduces to a long exact sequence as $H^i(I, \mathbb{Z}/p) = 0$ for $i \geq 2$, $H^0(I, \mathbb{Z}/p) = \mathbb{Z}/p$, $H^1(I, \mathbb{Z}/p) = \text{Hom}(I, \mathbb{Z}/p) = \mathbb{Z}/p$, which reads

$$\begin{aligned} \cdots \rightarrow H^i(\Gamma_{\kappa_\nu}, \mathbb{Z}/p) \rightarrow H^i(\Gamma_{K_\nu}, \mathbb{Z}/p) \rightarrow H^{i-1}(\Gamma_{\kappa_\nu}, \text{Hom}(\hat{\mathbb{Z}}, \mathbb{Z}/p)) \rightarrow \\ \rightarrow H^{i+1}(\Gamma_{\kappa_\nu}, \mathbb{Z}/p) \rightarrow H^{i+1}(\Gamma_{K_\nu}, \mathbb{Z}/p) \rightarrow \cdots \end{aligned}$$

and the fact that the extension splits implies that this long exact sequence breaks into short exact sequences

$$0 \rightarrow H^i(\Gamma_{\kappa_\nu}, \mathbb{Z}/p) \rightarrow H^i(\Gamma_{K_\nu}, \mathbb{Z}/p) \xrightarrow{r} H^{i-1}(\Gamma_{\kappa_\nu}, \mathbb{Z}/p)$$

where r can also be explicitly described in terms of cocycles, see[GMS], p.16.

In [B-P] the following theorem was proven (loc. cit, Theorem 5.1):

Theorem 1.1. *There is a natural isomorphism $H_s^*(\mathfrak{A}_{2n+1}, \mathbb{Z}/2\mathbb{Z}) \simeq H_s^*(\mathfrak{A}_{2n}, \mathbb{Z}/2\mathbb{Z})$, and as a $\mathbb{Z}/2\mathbb{Z}$ -vector space*

$$H_s^*(\mathfrak{A}_{2n}, \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{0 \leq i \leq n} \mathbb{Z}/2\mathbb{Z} \cdot w_{2i} \oplus \bigoplus_{0 < i \leq n} \mathbb{Z}/2\mathbb{Z} \cdot u_1 \wedge w_{2i}$$

where w_j are the (images in stable cohomology of) the Stiefel-Whitney classes in $H^j(\mathfrak{A}_{2n}, \mathbb{Z}/2\mathbb{Z})$ obtained from the cohomology ring of the real orthogonal group $O(2n)$ via the inclusions $\mathfrak{A}_{2n} \subset \mathfrak{S}_{2n} \subset O(2n)$. The class u_1 is a one-dimensional cohomology class which can be described as follows:

Putting $N = 2n$, the group \mathfrak{S}_N acts generically freely on the complement $\mathbb{C}^{N-1} - H$ of the braid hyperplane arrangement H in the standard permutation representation \mathbb{C}^{N-1} , and $(\mathbb{C}^{N-1} - H)/\mathfrak{S}_N \simeq \mathbb{C}^{N-1} - \Delta$, the complement of the discriminant. Taking a nonramified double covering $\widetilde{\mathbb{C}^{N-1} - \Delta}$ of $\mathbb{C}^{N-1} - \Delta$ corresponding to the inclusion $\mathfrak{A}_N \subset \mathfrak{S}_N$, one gets a description of u_1 as the generator of $H^1(\widetilde{\mathbb{C}^{N-1} - \Delta})$ given by the root of the discriminant.

In our Theorem 3.6 we determine $H_s^*(\mathfrak{A}_N, \mathbb{Z}/p\mathbb{Z})$ completely for odd primes p .

We base our approach to the computation of the stable cohomology of alternating groups on the following lemmas.

Lemma 1.2. *Suppose a group is a product $G \times A$ of finite groups G and A with A abelian. Then there is the Künneth decomposition*

$$H_s^n(G \times A, \mathbb{Z}/p) \simeq \bigoplus_{i+j=n} H_s^i(G, \mathbb{Z}/p) \otimes H_s^j(A, \mathbb{Z}/p).$$

Proof. It is known ([Bogo93], Lemma 7.1) that if we choose a free presentation $\pi : \mathbb{Z}^n \twoheadrightarrow A$ of A , then the kernel of the stabilization map coincides with the kernel of π^* . In other words, if one realizes \mathbb{Z}^n as the fundamental group of some algebraic torus T , with cover $T' \rightarrow T \simeq T'/A$ corresponding to A , and realizes T' as a maximal torus in some $\mathrm{GL}(W)$, then stabilization is achieved by considering the image of the cohomology of A in the cohomology of $T \simeq T'/A \subset W^L/A$. This is so because one can find a product of circles $(S^1)^m$ in the complement of any divisor D in $T \simeq (\mathbb{C}^*)^m$, the inclusion being a homotopy equivalence, so the cohomology of T is already stable. The product of circles can be found by induction on the dimension m of T ; if $m = 1$, one chooses a circle in the complex plane \mathbb{C} not passing through the finite number of points which D consists of. If $m > 1$, one views T as a subset of W , which is stratified into torus orbits of lower dimension. Each of these is isomorphic to an algebraic tori. Choose a codimension 1 torus orbit T_1 adjacent to T . The closure \bar{D} inside W of a divisor $D \subset T$ meets T_1 in a proper algebraic subset, and by the induction hypothesis there is a real submanifold $M \simeq (S^1)^{m-1}$ in the complement of $\bar{D} \cap T_1$. If x_1, \dots, x_m are coordinates in W such that $T = \{x_i \neq 0 \ \forall i\}$, $T_1 = \{x_1 = 0 \wedge x_j \neq 0 \ \forall j \neq 1\}$, then $W = W' \oplus \mathbb{C}$ where $W' = \{x_1 = 0\}$. If we choose a small circle $S_\epsilon \subset \mathbb{C}$, then $M \times S_\epsilon \subset W$ will be in a small neighbourhood of M hence will not intersect D .

This argument can be made relative: note first that there is always a natural surjection $H_s^*(G, \mathbb{Z}/p) \otimes H_s^*(A, \mathbb{Z}/p) \rightarrow H_s^*(G \times A, \mathbb{Z}/p)$ as the Zariski topology on a product is finer than the product topology, and to show it is an isomorphism, it suffices to note the following: suppose $T \simeq (\mathbb{C}^*)^m \simeq T'/A$ is as before, and V is a generically free G -representation, V^L the open part where the action is free. Then if $D \subset (V^L/G) \times T$ is any divisor, there is always a divisor $D' \subset V^L/G$

and a relatively compact subset $U^L/G \subset V^L/G - D'$ with a (trivial) iterated circle fibration $U^L/G \times (S^1)^m \subset ((V^L/G) \times T) - D$ such that $U^L/G \times (S^1)^m$ and $((V^L/G) \times T) - D$ are homotopy equivalent.

Indeed, viewing $V^L/G \times T \subset V^L/G \times W$, the latter being a vector bundle, we have a zero section $V^L/G \subset V^L/G \times W$. Moreover, $\bar{D} \cap V^L/G$, where \bar{D} is the closure of D in $V^L/G \times W$, will be contained in some divisor D' . Shrinking $V^L/G - D'$ slightly, we can find a relatively compact open subset $U^L/G \subset V^L/G - D'$ homotopy equivalent to $V^L/G - D'$ and with the claimed circle fibration. \square

We say that the stable cohomology $H_s^*(G, \mathbb{Z}/p)$ is detected by abelian subgroups if the map induced by the restriction to abelian subgroups

$$H_s^*(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \prod_A H_s^*(A, \mathbb{Z}/p\mathbb{Z})$$

is injective (where A ranges over all abelian subgroups of G). We will then also use the following principle which follows from Lemma 1.2.

Lemma 1.3. *Suppose G_1 and G_2 are finite groups such that at least one of $H_s^*(G_1, \mathbb{Z}/p)$ or $H_s^*(G_2, \mathbb{Z}/p)$ is detected by abelian subgroups. Then one has a Künneth formula in stable cohomology*

$$H_s^*(G_1 \times G_2, \mathbb{Z}/p) \simeq H_s^*(G_1, \mathbb{Z}/p) \otimes H_s^*(G_2, \mathbb{Z}/p).$$

Proof. There is always the natural surjection

$$H_s^*(G_1, \mathbb{Z}/p) \otimes H_s^*(G_2, \mathbb{Z}/p) \xrightarrow{p} H_s^*(G_1 \times G_2, \mathbb{Z}/p).$$

Without loss of generality, we can assume that abelian subgroups A_i are a detecting family for the stable cohomology of G_1 :

$$H_s^*(G_1, \mathbb{Z}/p) \hookrightarrow \prod_{i \in I} H_s^*(A_i, \mathbb{Z}/p).$$

Now by Lemma 1.2, there is an injection

$$H_s^*(G_1, \mathbb{Z}/p) \otimes H_s^*(G_2, \mathbb{Z}/p) \xhookrightarrow{i} \prod_i H_s^*(A_i \times G_2, \mathbb{Z}/p).$$

But $i = (\prod \text{res}_{A_i \times G_2}) \circ p$ where

$$\prod \text{res}_{A_i \times G_2} : H_s^*(G_1 \times G_2, \mathbb{Z}/p) \rightarrow \prod_i H_s^*(A_i \times G_2, \mathbb{Z}/p)$$

is the product of restriction maps. Hence p is also injective. \square

Lemma 1.4. *Let G be a finite group such that $H_{\text{nr}}^i(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i > 0$. Then every stable class $a \in H_s^*(G, \mathbb{Z}/p\mathbb{Z})$ is nontrivial on the centralizer $C(g)$ of some element $g \in G$, i.e. the restriction $\text{res} : H_s^*(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_s^*(C(g), \mathbb{Z}/p\mathbb{Z})$ is nonzero.*

Proof. With the notation established above we have the maps of groups

$$I \subset \text{Gal}(K_\nu) \subset \text{Gal}(K) \twoheadrightarrow G,$$

and the image of the inertia subgroup I in G is cyclic, generated by g say, and the image of the decomposition group $\text{Gal}(K_\nu)$ in G belongs to the centralizer $C(g)$. As the residue map ∂_ν factors over $\text{Gal}(K_\nu)$, we obtain the assertion. \square

Recall the exact sequence

$$1 \rightarrow I \rightarrow \Gamma_{K_\nu} \rightarrow \Gamma_{\kappa_\nu} \rightarrow 1$$

where I is the inertia subgroup of the decomposition group Γ_{K_ν} associated to the valuation ν of $K = \mathbb{C}(V)^G$. The following Lemma allows one to increase the usefulness of Lemma 1.4 in inductive arguments further.

Lemma 1.5. *Let G be a finite group and let $a \in H_s^*(G, \mathbb{Z}/p)$ be a stable class. For ν a divisorial valuation of K , the image of the topologically cyclic inertia subgroup I in G is cyclic, generated by h say. There is a natural class $d_\nu(a) \in H_s^{n-1}(Z_G(h), \mathbb{Z}/p)$ such that the residue $\partial_\nu(a) \in H^{n-1}(\Gamma_{\kappa_\nu}, \mathbb{Z}/p)$ is the pull-back of $d_\nu(a)$ to Γ_{κ_ν} via the maps $\Gamma_{\kappa_\nu} \subset \Gamma_{K_\nu} \simeq I \oplus \Gamma_{\kappa_\nu} \rightarrow Z(h)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & I & \longrightarrow & \Gamma_{K_\nu} & \longrightarrow & \Gamma_{\kappa_\nu} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \langle h \rangle & \xrightarrow{f} & Z(h) & \xrightarrow{g} & Z(h)/\langle h \rangle \longrightarrow 1 \\
 & & \uparrow a & & \uparrow b & & \uparrow c \\
 1 & \longrightarrow & \langle h \rangle & \longrightarrow & \langle h \rangle \times Z(h) & \longrightarrow & Z(h) \longrightarrow 1
 \end{array}$$

where the arrow a is the identity, b is the couple $(i_{\langle h \rangle}, \text{id}_{Z(h)})$ where $i_{\langle h \rangle} : \langle h \rangle \rightarrow Z(h)$ is the inclusion, and c is the projection. Here f and g are defined as follows: the extension defining Γ_{K_ν} splits, $\Gamma_{K_\nu} \simeq \hat{\mathbb{Z}} \oplus \Gamma_{\kappa_\nu}$, so the map $\Gamma_{\kappa_\nu} \rightarrow Z(h)/\langle h \rangle$ lifts to a map $g_1 : \Gamma_{\kappa_\nu} \rightarrow Z(h)$. The map g is simply $(1, g_1)$. The map f is $(f_1, 1)$ where $f_1 : I \rightarrow \langle h \rangle$ is the natural map. As $\hat{\mathbb{Z}}$ has cohomological dimension one, $H^i(\Gamma_{K_\nu}, \mathbb{Z}/p) \simeq H^{i-1}(\Gamma_{\kappa_\nu}, \mathbb{Z}/p) \oplus H^i(\Gamma_{\kappa_\nu}, \mathbb{Z}/p)$ and the residue map ∂_ν was defined by the restriction of a to Γ_{K_ν} and projecting to $H^{i-1}(\Gamma_{\kappa_\nu}, \mathbb{Z}/p)$. By the commutativity of the diagram, we may thus define a class $d_\nu(a)$ with the requested properties as follows: we restrict $a \in H_s^n(G, \mathbb{Z}/p)$ to $Z(h)$, and then take the pull-back

$b^*(\text{res}_{Z(h)}^G(a)) \in H_s^n(\langle h \rangle \times Z(h), \mathbb{Z}/p)$ and project this unto the component $H_s^{n-1}(Z(h), \mathbb{Z}/p)$ in the Künneth decomposition, using Lemma 1.2. This defines $d_\nu(a) \in H_s^{n-1}(Z(h), \mathbb{Z}/p)$. \square

Define a subgroup H of G recursively to be an *iterated centralizer* if it is the centralizer of an element in G or a centralizer of an element inside another iterated centralizer.

Corollary 1.6. *Assume that G is such that each iterated centralizer has trivial unramified cohomology. Then any element $a \in H_s^n(G, \mathbb{Z}/p)$ is nontrivial on some abelian p -subgroup.*

Proof. We use induction on the cohomological degree, hence assume that every element in $H_s^i(H, \mathbb{Z}/p)$, for all $i < n$, and for all iterated centralizers H in G , is nontrivial on some abelian subgroup. By assumption, we get from Lemma 1.5 that $d_\nu(a) \in H_s^{n-1}(Z(h), \mathbb{Z}/p)$ is nontrivial for some h . Hence $d_\nu(a)$ is nontrivial on some abelian p -subgroup A of $Z(h)$. By the construction of $d_\nu(a)$ in Lemma 1.5 we have that a will then be nontrivial when restricted to $H_s^n(\langle h, A \rangle, \mathbb{Z}/p)$ where $\langle h, A \rangle$ is the abelian subgroup of G generated by h and A . \square

The Steenrod power operations Sq^i, P^j (see [Steen], [A-M] II.2) are natural transformations

$$Sq^i : H^j(X, \mathbb{Z}/2) \rightarrow H^{j+i}(X, \mathbb{Z}/2),$$

$$P^i : H^j(X, \mathbb{Z}/p) \rightarrow H^{j+2i(p-1)}(X, \mathbb{Z}/p), \quad p \text{ an odd prime,}$$

on the category of CW-complexes with continuous maps $f : X \rightarrow Y$. By functoriality, applied to the map $BG \rightarrow (V^L/G) - D$, where D is some divisor, Sq^i, P^j induce operations on $H_s^*(G, \mathbb{Z}/p)$.

For later use, we recall here the structure theorem for the cohomology of wreath products due to Steenrod [Steen], Section VII, see also [Mann78], Theorem 3.1 and [A-M], IV. 4, Theorem 4.1. We suppress the $\mathbb{Z}/p\mathbb{Z}$ -coefficients in cohomology groups now, i.e. write $H^*(X)$ for $H^*(X, \mathbb{Z}/p\mathbb{Z})$.

Theorem 1.7. *Let H be a group, and let $H \wr \mathbb{Z}/p = (H)^p \rtimes \mathbb{Z}/p$ be the wreath product where $\mathbb{Z}/p\mathbb{Z}$ acts by cyclically permuting the copies of H . Let $\text{id} \times \Delta^p : \mathbb{Z}/p\mathbb{Z} \times H \rightarrow \mathbb{Z}/p \rtimes (H)^p$ be the inclusion $(\text{id} \times \Delta^p)(z, a) = (z; (a, \dots, a))$ (p -times a) and denote by $t : H^*(H^p) \rightarrow H^*(H \wr \mathbb{Z}/p)$ the transfer. Then the sequence*

$$H^*(H^p) \xrightarrow{t} H^*(H \wr \mathbb{Z}/p) \xrightarrow{(\text{id} \times \Delta^p)^*} H^*(\mathbb{Z}/p\mathbb{Z} \times H)$$

is exact.

Moreover, for any $u \in H^j(H)$ there is a class $P(u) \in H^{jp}(H \wr \mathbb{Z}/p)$ (constructed by Steenrod) such that

- (i) If $j : H^p \rightarrow \mathbb{Z}/p\mathbb{Z} \ltimes (H)^p$ is the natural inclusion, then $j^*(P(u)) = u^{\otimes p}$.
- (ii) In the Künneth decomposition of $(\text{id} \times \Delta^p)^*(P(u))$ in $H^*(\mathbb{Z}/p\mathbb{Z} \times H)$ we have

$$(\text{id} \times \Delta^p)^*(P(u)) = \sum w_k \otimes D_k(u)$$

where w_k is a generator of $H^k(\mathbb{Z}/p\mathbb{Z})$ and $D_k : H^q(H) \rightarrow H^{p-q-k}(H)$ are homomorphisms which satisfy

(iii)

$$\beta D_{2k}(u) = D_{2k-1}(u), \beta D_{2k-1}(u) = 0, \beta D_0(u) = 0$$

where β is the Bockstein homomorphism, i.e. connecting homomorphism in the long exact sequence coming from the short exact sequence

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0.$$

The maps D_k are used originally by Steenrod to define the Steenrod powers P^i , hence the Hopf algebras (Steenrod algebras) $\mathcal{A}(p)$. More precisely,

$$P^i(u) = (-1)^{i+mj(j+1)/2} (m!)^j D_{(p-1)(j-2i)}(u) \in H^{j+2(p-1)i}(H, \mathbb{Z}/p\mathbb{Z}).$$

In this setting, any $c \in H^*(H \wr \mathbb{Z}/p)$ can be written as

$$c = t(c_1) + c_2 \cdot P(c_3)$$

with $c_1 \in H^*(H^p)$, $c_2 \in H^*(\mathbb{Z}/p\mathbb{Z})$ and $c_3 \in H^*(H)$.

Here we view $H^*(H \wr \mathbb{Z}/p)$ as a module over $H^*(\mathbb{Z}/p\mathbb{Z})$ via the cohomology pull-back induced by the surjection $H \wr \mathbb{Z}/p \rightarrow \mathbb{Z}/p\mathbb{Z}$. The Steenrod operations P^i have the following formal properties:

- (1) $P^0 = \text{id}$.
- (2) If $\dim(x) = 2n$, then $P^n(x) = x^p$.
- (3) If $2i > \dim(x)$, then $P^i(x) = 0$.
- (4) (Cartan formula) $P^i(x \cup y) = \sum_{j=0}^i P^j(x) \cup P^{i-j}(y)$.

A consequence of the Bloch-Kato conjecture (now a theorem by the work of Voevodsky, Rost and many others) is

Lemma 1.8. *The Steenrod cohomology operations Sq^i , P^j are zero in stable cohomology $H_s^*(G, \mathbb{Z}/p\mathbb{Z})$.*

Proof. Any stable class $a \in H_s^*(G, \mathbb{Z}/p\mathbb{Z}) \subset H^*(\text{Gal}(K), \mathbb{Z}/p)$ arises as the pull-back from some torus $T \simeq (\mathbb{C}^*)^m$, more precisely, a has a representative a' in the cohomology of $H^*((V^L/G) - D, \mathbb{Z}/p)$ and there is a regular map $f : (V^L/G) - D \rightarrow T$ together with a class $\hat{a} \in H^*(T, \mathbb{Z}/p)$ with $f^*(\hat{a}) = a'$. This follows from the fact, which is a consequence of the Bloch-Kato conjecture, that $H^*(\text{Gal}^{\text{ab}}(K), \mathbb{Z}/p) \rightarrow H^*(\text{Gal}(K), \mathbb{Z}/p)$ is surjective where $\text{Gal}^{\text{ab}}(K)$ is the abelianized Galois group $\text{Gal}(K)/[\text{Gal}(K), \text{Gal}(K)]$.

Since the Steenrod power operations are trivial in the cohomology algebra of the torus T (which is an exterior algebra on one-dimensional generators), the assertion of the Lemma will then follow from the functoriality of the cohomology operations.

It remains to explain in some more detail how from the surjection $H^*(\text{Gal}^{\text{ab}}(K), \mathbb{Z}/p) \rightarrow H^*(\text{Gal}(K), \mathbb{Z}/p)$ we get the map $f : (V^L/G) - D \rightarrow T$. There is a finite abelian quotient $\text{Gal}(K) \twoheadrightarrow A$ such that a is induced from a class a'' in the cohomology of A . The group A corresponds to an unramified abelian covering \tilde{X} of some nonempty open affine subvariety $X \subset V^L/G$. The coordinate ring $\mathbb{C}[\tilde{X}]$ contains an arbitrary finite dimensional representation of A as the regular functions on an A -orbit in \tilde{X} are precisely the regular representation $\mathbb{C}[A]$ and this is also a subrepresentation (not only a quotient) because A is reductive. In particular, embedding A in a torus of diagonal matrices in some $\text{GL}_m(\mathbb{C})$, one obtains a dominant regular map from an open subset X' of $X = \tilde{X}/A$, hence some $(V^L/G) - D$ to a torus $T = (\mathbb{C}^*)^L/A \simeq (\mathbb{C}^*)^m$ for which it holds by construction that the image of the class a'' in the cohomology of T induces a representative $a' \in H^*((V^L/G) - D, \mathbb{Z}/p)$ of a . \square

Thus the techniques used in the present article are mainly topological in flavour; for the connection to motivic cohomology and further developments the reader may consult [Kahn-Su00], [Kahn11], [Ngu1], [Ngu2], [TY11].

2. DETECTION BY ELEMENTARY ABELIAN p -SUBGROUPS

In this section we want to prove the following Theorem.

Theorem 2.1. *Let p as always be an odd prime and \mathfrak{A}_N the alternating group on N letters. Then $H_s^*(\mathfrak{A}_N, \mathbb{Z}/p)$ is detected by elementary abelian p -subgroups.*

We denote by $G_n = \mathbb{Z}/p\mathbb{Z} \wr \cdots \wr \mathbb{Z}/p\mathbb{Z}$ (n factors) the iterated wreath product of n cyclic p -groups. This is the p -Sylow of \mathfrak{A}_{p^n} . If N is

arbitrary (not necessarily a power of p) expand it in base p :

$$N = a_0 + a_1p + \cdots + a_mp^m$$

with $0 \leq a_j < p$, $a_m \neq 0$, and note that this gives rise to a natural inclusion

$$i_{a_1, \dots, a_m} : \mathfrak{A}_{a_1, \dots, a_m} := \prod_1^{a_1} \mathfrak{A}_p \times \prod_1^{a_2} \mathfrak{A}_{p^2} \times \cdots \times \prod_1^{a_m} \mathfrak{A}_{p^m} \hookrightarrow \mathfrak{A}_N$$

and that a p -Sylow subgroup in \mathfrak{A}_N is given by the product of p -Sylow subgroups in the factors in $\mathfrak{A}_{a_1, \dots, a_m}$. Hence it follows from Lemma 1.3 that it is enough to prove Theorem 2.1 for $N = p^n$.

First we will prove the weaker

Theorem 2.2. *The stable cohomology $H_s^*(\mathfrak{A}_N, \mathbb{Z}/p)$ is detected by abelian p -subgroups.*

The proof of Theorem 2.2 will follow from the structures of centralizers in *complete monomial groups*.

Definition 2.3. Let H be a group. The *complete monomial group* of degree m on H is the group $\Sigma_m(H) := H \wr \mathfrak{S}_m = (H)^m \rtimes \mathfrak{S}_m$, where \mathfrak{S}_m is the symmetric group on m letters.

A *monomial cycle* in $\Sigma_m(H)$ is an element of the form $((h_1, \dots, h_m); \sigma)$ where $\sigma \in \mathfrak{S}_m$ is a cycle. The *determinant class* of a monomial cycle is the conjugacy class in H of the product $h_1 \cdot \dots \cdot h_m$. The *length* of a monomial cycle is the length of the underlying cycle σ .

We have to recall some results from [Ore] on the structure of conjugacy classes and centralizers in groups $\Sigma_m(H)$:

- (1) Two monomial cycles in $\Sigma_m(H)$ are conjugate if and only if they have the same length and determinant class ([Ore], Theorem 6).
- (2) Any element $x \in \Sigma_m(H)$ can be written uniquely as a product of commuting monomial cycles (the underlying cycles in \mathfrak{S}_m have no common variables) ([Ore], Theorem 3).
- (3) From (1) and (2) follows the description of conjugacy classes in $\Sigma_m(H)$: two elements x', x in $\Sigma_m(H)$ are conjugate if the monomial cycles in their decompositions in (2) can be matched in such a way that corresponding cycles have the same length and determinant class.
- (4) Let $c = ((h, \dots, 1, 1); \sigma)$ be a monomial cycle of length m in $\Sigma_m(H)$. Then its centralizer in $\Sigma_m(H)$ is an extension

$$1 \rightarrow Z_H(h) \rightarrow Z_{\Sigma_m(H)}(c) \rightarrow \mathbb{Z}/m \rightarrow 0$$

([Ore], p.20, 21). In other words, the centralizer $Z_{\Sigma_m(H)}(c)$ is generated by c and the group $Z_H(h)$ embedded diagonally into $H^m \subset \Sigma_m(H)$.

(5) ([Ore], Theorem 8): Let

$$x = x_1 \cdot \dots \cdot x_l, \quad x_i = y_1^{(i)} \cdot \dots \cdot y_{k_i}^{(i)}$$

be the decomposition of an element x in $\Sigma_m(H)$ into disjoint monomial cycles $y_j^{(i)}$ as in (2), where we group the cycles of equal length and determinant class together: for fixed i , all $y_j^{(i)}$ have determinant class h_i and length n_i . Then the centralizer of x in $\Sigma_m(H)$ has a description as

$$Z_{\Sigma_m(H)}(x) = \prod_{i=1}^l \Sigma_{k_i}(Z_{\Sigma_{n_i}(H)}(y_1^{(i)}))$$

where as in (4) the centralizer of $y_1^{(i)}$ (or any $y_j^{(i)}$) in the group $\Sigma_{n_i}(H)$ is an extension

$$1 \rightarrow Z_H(h_i) \rightarrow Z_{\Sigma_{n_i}(H)}(y_1^{(i)}) \rightarrow \mathbb{Z}/n_i\mathbb{Z} \rightarrow 0.$$

From the last fact (5) we get immediately

Lemma 2.4. *Let $\Sigma_m(A)$ be a complete monomial group with A abelian. Then the centralizer of any element x in $\Sigma_m(A)$ is a product of groups of the same type*

$$Z_{\Sigma_m(A)}(x) = \prod_{h=1}^M \Sigma_{k_h}(A_h)$$

where the A_h are abelian.

Proof. It suffices to remark that any central extension of a cyclic group by an abelian group is again abelian. \square

To prove Theorem 2.2 it suffices, by the technique of Lemma 1.5 exposed above, to show the following.

Lemma 2.5. *Let p be an odd prime and let x be an element of order a power of p in a group $A \wr \mathfrak{A}_m = A^m \rtimes \mathfrak{A}_m$ where A is an abelian p -group. Then there is a group*

$$Z' = \prod_{h=1}^M A_h \wr \mathfrak{A}_{k_h}$$

where all the A_h are abelian p -groups, and with the property that Z' is contained in the centralizer $Z_{A \wr \mathfrak{A}_m}(x)$ and contains a p -Sylow of $Z_{A \wr \mathfrak{A}_m}(x)$.

Proof. We consider the group $A \wr \mathfrak{A}_m$ as a subgroup of the complete monomial group $\Sigma_m(A)$. By Lemma 2.4 it suffices to determine the intersection of $Z_{\Sigma_m(A)}(x) = \prod_{h=1}^M \Sigma_{k_h}(A_h)$ and $A \wr \mathfrak{A}_m$. As p is odd, it is clear that the intersection contains

$$Z' = \prod_{h=1}^M A_h \wr \mathfrak{A}_{k_h}$$

and that the complete centralizer $Z_{A \wr \mathfrak{A}_m}(x)$ is an extension

$$1 \rightarrow Z' \rightarrow Z_{A \wr \mathfrak{A}_m}(x) \rightarrow (\mathbb{Z}/2)^r \rightarrow 0$$

where $(\mathbb{Z}/2)^r$ is an elementary abelian 2-group which can be identified with the kernel of the sign

$$\prod_{h=1}^M \mathfrak{S}_{k_h} \subset \mathfrak{S}_{\sum k_h} \rightarrow \{\pm 1\}$$

modulo the subgroup $\prod_{h=1}^M \mathfrak{A}_{k_h}$. The statement follows as p is odd. \square

Thus we obtain

Proof. (of Theorem 2.2) We will prove more generally that $H_s^*(G, \mathbb{Z}/p)$ is detected by abelian p -subgroups where G is any group which is a product of groups $A \wr \mathfrak{A}_m$ with A an abelian p -group. We have that $H_s^*(G, \mathbb{Z}/p)$ is detected by $H_s^*(\text{Syl}_p(G), \mathbb{Z}/p)$, and the higher unramified cohomology of $\text{Syl}_p(G)$ is trivial. This follows immediately from [B-P], Lemma 2.4, namely, if one forms a wreath product of groups, each of which has stably rational generically free linear quotients, then the wreath product inherits this property.

Hence every element a in $H_s^n(G, \mathbb{Z}/p)$ will, in the notation of Lemma 1.5, give a nontrivial $d_\nu(a) \in H_s^{n-1}(Z_G(h), \mathbb{Z}/p)$ for some element $h \in G$ of p -power order. Thus it will be enough to show that $H_s^{n-1}(Z_G(h), \mathbb{Z}/p)$ is detected by abelian p -subgroups. But $H_s^{n-1}(Z_G(h), \mathbb{Z}/p)$ is detected by $H_s^{n-1}(\text{Syl}_p(Z_G(h)), \mathbb{Z}/p)$ and, by Lemma 2.5, $\text{Syl}_p(Z_G(h))$ is contained in a group which in turn is contained in $Z_G(h)$ and is again a product of groups of type $A \wr \mathfrak{A}_m$. Hence we can conclude by induction on the cohomological degree n . \square

Now we prove Theorem 2.1. It will follow immediately from

Proposition 2.6. *Let $N = p^n$, and suppose that A is an abelian p -subgroup of \mathfrak{A}_N . Thus one can write*

$$A = \prod_{i=1}^k (\mathbb{Z}/(p^{l_i}))^{r_i}, \quad l_i, r_i \in \mathbb{N}.$$

If A is not reduced to a single cyclic group $\mathbb{Z}/(p^l)$, then A is contained in a product of alternating groups $\prod_{j=1}^h \mathfrak{A}_{t_h} \subset \mathfrak{A}_N$ with $t_h < N$ for all h .

Once we have this Proposition, the proof of Theorem 2.1 is an induction: it suffices to prove it for $N = p^n$, and we may assume that detection by elementary abelian subgroups holds for the stable cohomology of all \mathfrak{A}_j with $j < N$. Now clearly, $H_s^1(\mathfrak{A}_N, \mathbb{Z}/p)$ is detected by elementary abelian p -subgroups, for $H^1(\mathfrak{A}_N, \mathbb{Z}/p)$ can be identified with characters $\chi : \mathfrak{A}_N \rightarrow \mathbb{Z}/p$ whence $H_s^1(\mathfrak{A}_N, \mathbb{Z}/p) = 0$ unless $p = 3$ and $N = 3$ so that $\mathfrak{A}_N = \mathbb{Z}/3$. So we have to show that any stable class $a \in H_s^i(\mathfrak{A}_N, \mathbb{Z}/p)$ for $i \geq 2$ is nontrivial on an elementary abelian p -subgroup. By Theorem 2.2 a is nontrivial on an abelian p -subgroup A with $\text{rk}(A/pA) \geq 2$ (as the stable cohomology of A is an exterior algebra on $\text{rk}(A/pA)$ generators). Such an A is contained in a product of smaller alternating groups by Proposition 2.6. Thus the proof is complete by induction.

Proof. (of Proposition 2.6)

Denote by $X_N = \{1, \dots, N\}$ the set of letters on which the ambient $\mathfrak{S}_N \supset A$ acts. Let

$$X_N = \coprod_{\alpha} X_{N, \alpha}$$

be the decomposition of X_N into A -orbits. Let $X_{N, \alpha_0} =: X$ be a fixed orbit. This orbit is isomorphic to a quotient \bar{A} of A , hence a group of the same form

$$\bar{A} = \prod_{i=1}^{\bar{k}} (\mathbb{Z}/(p^{\bar{l}_i}))^{\bar{r}_i}$$

and the action of A on this orbit is via *the regular representation* of \bar{A} on itself. In other words, A embeds into a subgroup $\prod_{\alpha} \bar{A}_{\alpha}$ of \mathfrak{A}_N where each \bar{A}_{α} is embedded into a subgroup $\mathfrak{A}_{\text{ord}(\bar{A}_{\alpha})}$ via the regular representation.

In summary, it suffices to prove the statement of Proposition 2.6 for the case that the group A in its statement is embedded into the ambient \mathfrak{A}_N via the regular representation. We can write $A = A' \times \mathbb{Z}/(p^k)$ with $\text{rk}(A'/pA') < \text{rk}(A/pA)$. Moreover, by definition of the regular representation, the composition of arrows

$$A = A' \times \mathbb{Z}/(p^k) \hookrightarrow \mathfrak{A}_{|A'|} \times \mathbb{Z}/(p^k) \hookrightarrow \mathfrak{A}_{|A'|} \wr \mathbb{Z}/(p^k) \hookrightarrow \mathfrak{A}_{|A'| \cdot p^k} \simeq \mathfrak{A}_{|A|}$$

gives the regular representation of A where the first arrow \hookrightarrow from the left is induced by the regular representation of A' , the second such arrow embeds $\mathfrak{A}_{|A'|} \times \mathbb{Z}/(p^k)$ into the wreath product $\mathfrak{A}_{|A'|} \wr \mathbb{Z}/(p^k)$ by

sending $(a; \sigma)$ to $(a, a, \dots, a; \sigma)$ as usual, and the last arrow embeds the wreath product $\mathfrak{A}_{|A'|} \wr \mathbb{Z}/(p^k)$ into $\mathfrak{A}_{|A'| \cdot p^k}$ by partitioning the set of $|A'| \cdot p^k$ objects which $\mathfrak{A}_{|A'| \cdot p^k}$ permutes into p^k disjoint groups of $|A'|$ objects, and letting $\mathbb{Z}/(p^k)$ act by cyclically rotating these groups, and letting $(\mathfrak{A}_{|A'|})^{p^k}$ act via permutations within these groups. It follows that

$$A \subset \mathfrak{A}_{|A'|} \times \mathfrak{A}_{p^k}$$

where now \mathfrak{A}_{p^k} is embedded into $\mathfrak{A}_{|A|}$ as arbitrary alternating (not only cyclic) permutations of the p^k groups of items. Note that elements of the two subgroups \mathfrak{A}_{p^k} and $\mathfrak{A}_{|A'|}$ of the group $\mathfrak{A}_{|A|}$ commute, and the two subgroups intersect trivially, so that we do have a direct product. Moreover, if A is not reduced to a single cyclic group, we have that A' is not the trivial group, and $p^k < |A|$. \square

3. STABLE COHOMOLOGY OF ALTERNATING GROUPS

Let \mathfrak{A}_n be, as in the previous section, the alternating group on n letters, and let p be an odd prime (the case $p = 2$ has been treated in [B-P]). We assume first $n = p^m$ for simplicity.

We have to know the way elementary abelian p -subgroups sit inside \mathfrak{A}_n for the following. We summarize everything in the following Lemma which is proven by arguments analogous to those already used in the proof of Proposition 2.6.

Lemma 3.1. *Suppose $n = p^m$ and denote by $I_m := \{\underline{i} = (i_1, \dots, i_m) \in \mathbb{N}^m\}$ the set of all nonnegative integer sequences \underline{i} with*

$$p^m = i_1 p + i_2 p^2 + \dots + i_m p^m = \sum_{j=1}^m i_j p^j.$$

Then there is a natural bijection between I_m and the set of conjugacy classes of maximal elementary abelian p -subgroups in \mathfrak{S}_{p^m} . The subgroup $T(i_1, \dots, i_m)$ corresponding to \underline{i} can be described as follows: partition the set of integers $X = \{1, \dots, n\}$ into segments of p power lengths according to \underline{i} :

$$X = \bigcup_{j=1}^m \bigcup_{s=1}^{i_j} X_s^j$$

where X_s^j is a set with p^j elements,

$$X_s^j = \{i_1 p + \dots + i_{j-1} p^{j-1} + (s-1)p^j, \dots, i_1 p + \dots + i_{j-1} p^{j-1} + sp^j\}$$

for definiteness. The subset X_s^j corresponds to a subgroup $\mathfrak{S}_{p^j} = (\mathfrak{S}_{p^j})^{X_s^j} \subset \mathfrak{S}_{p^m}$ fixing all elements in X outside X_s^j . Inside $(\mathfrak{S}_{p^j})^{X_s^j}$ there is a copy of $(\mathbb{Z}/p\mathbb{Z})^j$, which we denote by $((\mathbb{Z}/p\mathbb{Z})^j)^{X_s^j}$, embedded via the regular representation, i.e. we identify the elements in X_s^j with the elements of $((\mathbb{Z}/p\mathbb{Z})^j)^{X_s^j}$ and the permutation action is then given by left multiplication.

We denote $T(0, \dots, p^{m-k}, \dots, 0)$ (a single nonzero entry p^{m-k} in the k -th place) by $T_{k,m}$.

Hence every maximal elementary abelian p -subgroup in \mathfrak{A}_n is conjugate -in \mathfrak{S}_n or \mathfrak{A}_n , it is the same thing- to one contained in $\mathfrak{A}_{p^{n-1}} \times \dots \times \mathfrak{A}_{p^{n-1}}$ (p factors) or conjugate to $T_{m,m}$.

The proof is immediate if one notices that under the action of some elementary abelian p -subgroup A the set X breaks up into A orbits of cardinality a p power, and the action of A restricted to an orbit embeds A into the permutation group of the elements of the orbit in such a way that the image is conjugate to the image of the regular representation. The result is in [A-M] VI. 1, Thm. 1.3, but also [Mui], Chapter II, §2, where it is ascribed to Dixon. For the statement that the conjugacy classes of maximal elementary abelian p -subgroups in \mathfrak{A}_n are the same as in \mathfrak{S}_n one can appeal to the following Lemma which we will also use in other instances below (it is e.g. in [Mann85], p. 269).

Lemma 3.2. *For $n = p^m$ the Weyl groups $W_{\mathfrak{S}_n}(T_{m,m}) = N_{\mathfrak{S}_n}(T_{m,m})/T_{m,m}$ resp. $W_{\mathfrak{A}_n}(T_{m,m})$ of $T_{m,m} \simeq (\mathbb{Z}/p\mathbb{Z})^m$ inside \mathfrak{S}_n resp. \mathfrak{A}_n are*

$$W_{\mathfrak{S}_n}(T_{m,m}) = \mathrm{GL}_m(\mathbb{F}_p), \quad W_{\mathfrak{A}_n}(T_{m,m}) = \mathrm{GL}_m^+(\mathbb{F}_p)$$

where $\mathrm{GL}_m^+(\mathbb{F}_p)$ is the kernel of the map $\mathrm{GL}_m(\mathbb{F}_p) \rightarrow \mathbb{Z}/2\mathbb{Z}$ given by the determinant raised to the power $(p-1)/2$.

In fact it is true that the Weyl group of any group H in the embedding $H \hookrightarrow \mathfrak{S}_{|H|}$ given by the regular representation is the group of outer automorphisms of H , which become all inner in $\mathfrak{S}_{|H|}$. Both statements of the Lemma follow from this remark as $\mathrm{Aut}((\mathbb{Z}/p\mathbb{Z})^m) = \mathrm{GL}(m, \mathbb{F}_p)$. Likewise, Lemma 3.2 implies that in the normalizer of any maximal elementary abelian p -subgroup in \mathfrak{S}_n there are elements which do not lie in \mathfrak{A}_n . Hence conjugacy classes of these in the two groups coincide.

We will also use in an essential way the Cárdenas-Kuhn Theorem to calculate the stable cohomology of \mathfrak{A}_n , so we recall the precise statement (see [A-M] III.5 for the proof).

Theorem 3.3. *Let $E \subsetneq S \subsetneq G$ be a closed system of finite groups, where the closedness means that every subgroup of S which is conjugate to E in G is already conjugate to E in S . Let $W_G(E) = N_G(E)/E$ resp. $W_S(E) = N_S(E)/E$ be the Weyl groups of E in G resp. S , and suppose that E is p -elementary and that $W_S(E)$ contains a p -Sylow of $W_G(E)$. Then the image of the restriction map*

$$\text{res}_E^G : H^*(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z})$$

is equal to

$$\text{im}(\text{res}_E^S : H^*(S, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z})) \cap H^*(E, \mathbb{Z}/p\mathbb{Z})^{W_G(E)}.$$

We will mostly use this in the form of the following

Corollary 3.4. *Let S be a p -Sylow of a finite group G , and let E be an elementary abelian p -subgroup of S . Suppose that any subgroup of S conjugate to E in G is conjugate to E in S . Then we have*

$$\begin{aligned} & \text{im}(\text{res}_E^G : H^*(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z})) \\ &= \text{im}(\text{res}_E^S : H^*(S, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z})) \cap H^*(E, \mathbb{Z}/p\mathbb{Z})^{W_G(E)}. \end{aligned}$$

Proof. It suffices to remark that $[G : S] \equiv [N_G(E) : N_S(E)] \not\equiv 0 \pmod{p}$. \square

Now if n is arbitrary (not necessarily a power of p), to understand $H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z})$, expand n in base p :

$$n = a_0 + a_1p + \cdots + a_mp^m$$

with $0 \leq a_j < p$, $a_m \neq 0$, and note that this gives rise to a natural inclusion

$$i_{a_1, \dots, a_m} : \mathfrak{A}_{a_1, \dots, a_m} := \prod_1^{a_1} \mathfrak{A}_p \times \prod_1^{a_2} \mathfrak{A}_{p^2} \times \cdots \times \prod_1^{a_m} \mathfrak{A}_{p^m} \hookrightarrow \mathfrak{A}_n$$

and that a p -Sylow subgroup in \mathfrak{A}_n is given by the product of p -Sylow subgroups in the factors in $\mathfrak{A}_{a_1, \dots, a_m}$. In the notation of Lemma 3.1, the group $\mathfrak{A}_{a_1, \dots, a_m}$ contains an elementary abelian p -subgroup

$$E := \prod_1^{a_1} T_{1,1} \times \prod_1^{a_2} T_{1,2} \times \cdots \times \prod_1^{a_m} T_{1,m} \simeq (\mathbb{Z}/p\mathbb{Z})^{\frac{n-a_0}{p}}.$$

Proposition 3.5. *The group E detects the stable cohomology of \mathfrak{A}_n , i.e.*

$$H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_s^*(E, \mathbb{Z}/p\mathbb{Z})$$

is injective.

Proof. It will be sufficient to prove this for $n = p^m$ as a Künneth theorem holds in stable cohomology for groups whose stable cohomology is detected by abelian subgroups, cf. Lemma 1.3. Now \mathfrak{A}_{p^m} contains the wreath product

$$\mathfrak{A}_{p^{m-1}} \wr \mathbb{Z}/p\mathbb{Z}$$

which detects the stable cohomology of \mathfrak{A}_{p^m} as it contains a p -Sylow. Using induction, it will be sufficient to prove that

$$H_s^*(\mathfrak{A}_{p^m}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_s^*(\mathfrak{A}_{p^{m-1}} \times \cdots \times \mathfrak{A}_{p^{m-1}})$$

is injective for $m > 1$. By Lemma 3.1 and because $H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z})$ is detected by elementary abelian p -subgroups, it will be sufficient to show that all positive-dimensional classes in $H^*(T_{m,m}, \mathbb{Z}/p\mathbb{Z})$ coming as restrictions from $H^*(\mathfrak{A}_{p^m}, \mathbb{Z}/p\mathbb{Z})$ are unstable. This follows from the calculation in [Mann85], Theorem 1.9, and the fact that the Bocksteins are zero in stable cohomology. \square

Theorem 3.6. *Let p be an odd prime as before. Then $H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) = 0$ in positive degrees unless $p = 3$. For $p = 3$ one has for $k \in \mathbb{N}$*

$$H_s^*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \simeq H_s^*(\mathfrak{A}_{3k+1}, \mathbb{Z}/3\mathbb{Z}), \quad H_s^d(\mathfrak{A}_{3k+2}, \mathbb{Z}/3\mathbb{Z}) = 0, \quad d > 0,$$

and

$$H_s^d(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \neq 0 \text{ for } d > 0 \iff d = k, \text{ and}$$

$$H_s^k(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \simeq \langle \det_k \rangle \text{ where } \text{res}_E^{\mathfrak{A}_{3k}}(\det_k) = e_1 \wedge \cdots \wedge e_k$$

where $H_s^*(E, \mathbb{Z}/3\mathbb{Z}) = H_s^*((\mathbb{Z}/3\mathbb{Z})^k, \mathbb{Z}/3\mathbb{Z})$ is an exterior algebra on one-dimensional generators e_1, \dots, e_k .

Basically, we would like to use the Cardenas-Kuhn Theorem with the elementary abelian subgroup E , and $S = \text{Syl}_p(\mathfrak{A}_n)$, $G = \mathfrak{A}_n$, but it will be more transparent to break it up into several steps.

Lemma 3.7. *For $p \neq 3$ an odd prime we have in positive degrees $H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) = 0$.*

Proof. The Weyl group $W_{\mathfrak{A}_n}(E)$ contains two obvious subgroups: (1) the group \mathfrak{A}_N permuting the $N := (n - a_0)/p$ copies of $\mathbb{Z}/p\mathbb{Z}$ in E , (2) a product $\prod_1^N (\mathbb{Z}/p\mathbb{Z})^{*,+}$ where $(\mathbb{Z}/p\mathbb{Z})^{*,+}$ is the subgroup of the group of units in $\mathbb{Z}/p\mathbb{Z}$ given as the kernel of $a \mapsto a^{(p-1)/2}$. The stable cohomology of E is an exterior algebra over $\mathbb{Z}/p\mathbb{Z}$ on N generators e_1, \dots, e_N . The \mathfrak{A}_N -invariants are concentrated in degrees 0, 1, $(N - 1)$, N , one-dimensional in each case and generated by

$$1, \quad e_1 + \cdots + e_N, \quad f_1 \wedge \cdots \wedge f_{N-1}, \quad e_1 \wedge \cdots \wedge e_N,$$

where f_1, \dots, f_{N-1} is a basis of the \mathfrak{A}_N -invariant complement to $e_1 + \dots + e_N$ in $H^1(E, \mathbb{Z}/p\mathbb{Z})$. All of these are not invariant under the scalings in $\prod_1^N (\mathbb{Z}/p\mathbb{Z})^{*,+}$ unless $p = 3$ when $(\mathbb{Z}/p\mathbb{Z})^{*,+}$ is reduced to $\{1\}$. \square

Lemma 3.8. *One has*

- (1) $H_s^d(\mathfrak{A}_{3k+2}, \mathbb{Z}/3\mathbb{Z}) = 0, d > 0.$
- (2) *There is a natural embedding*

$$H_s^*(\mathfrak{A}_{3k+1}, \mathbb{Z}/3\mathbb{Z}) \hookrightarrow H_s^*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}).$$

Proof. This is already contained in [B-P], Lemmas 4.1 and 4.2. For completeness, let us repeat the argument: the restriction $\text{res}_{\text{Syl}_3(\mathfrak{A}_{3k+2})}^{\mathfrak{A}_{3k+2}}$ factors through the restriction map induced from the embedding $\mathfrak{S}_{3k} \hookrightarrow \mathfrak{A}_{3k+2}$; but $H_s^*(\mathfrak{S}_{3k}, \mathbb{Z}/3\mathbb{Z}) = 0$ in positive degrees as the stable cohomology of \mathfrak{S}_{3k} is detected by its elementary abelian 2-subgroup generated by a maximal set of commuting transpositions. This proves (1), and (2) follows from the fact that the 3-Sylows in \mathfrak{A}_{3k} and \mathfrak{A}_{3k+1} are the same. \square

Lemma 3.9. *Let $n = 3k$ or $n = 3k + 1$. Then the Weyl group $N_{\mathfrak{A}_n}(E)$ of $E \simeq (\mathbb{Z}/3\mathbb{Z})^k$ in \mathfrak{A}_n sits in an extension*

$$1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^{k-1} \rightarrow W_{\mathfrak{A}_n}(E) \rightarrow \mathfrak{S}_k \rightarrow 1$$

where \mathfrak{S}_k acts by permuting the copies of $\mathbb{Z}/3\mathbb{Z}$ in $E \simeq (\mathbb{Z}/3\mathbb{Z})^k$, and the group $(\mathbb{Z}/2\mathbb{Z})^{k-1}$ acts by sending an even number of the generators g_i in the i th copy of $\mathbb{Z}/3\mathbb{Z}$ to their inverses g_i^{-1} . In stable cohomology $H_s^*(E, \mathbb{Z}/3\mathbb{Z}) = E(e_1, \dots, e_k)$ (exterior algebra), the action of the group $W_{\mathfrak{A}_n}(E)$ is generated by the (signed) transpositions sending e_i, e_j to $e_j, -e_i$, and transformations corresponding to elements in $(\mathbb{Z}/2\mathbb{Z})^{k-1}$ acting via sign changes $e_i \mapsto -e_i$ on an even number of the e_i .

Proof. Any element of the normalizer $N_{\mathfrak{A}_n}(E)$ induces a well-defined permutation of the copies of $\mathbb{Z}/3\mathbb{Z}$ in E . This gives a map to \mathfrak{S}_k which is *onto*: note that conjugating the 3-cycle (123) by $\tau = (23)$ exchanges the two nontrivial elements g, g^{-1} in $\mathbb{Z}/3\mathbb{Z} = \langle (123) \rangle$, so we can also transpose two copies of $\mathbb{Z}/3\mathbb{Z}$ in E by conjugating by an element in \mathfrak{A}_n . Now suppose $n \in N_{\mathfrak{A}_n}(E)$ induces the trivial element in \mathfrak{S}_k , so fixes all the copies of $\mathbb{Z}/3\mathbb{Z}$ in E (though not necessarily elementwise). Then the only possible nontrivial automorphism of each copy of $\mathbb{Z}/3\mathbb{Z}$ is exchanging g and g^{-1} as before. To conclude the proof, it suffices to note that if n induces the identity in $\text{Aut}((\mathbb{Z}/3\mathbb{Z})^k)$, then $n \in (\mathbb{Z}/3\mathbb{Z})^k$. \square

We can now turn to the

Proof. (of Theorem 3.6) The remaining assertion not covered by Lemma 3.7 and Lemma 3.8 are that

$$(1) \quad H_s^d(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \neq 0 \text{ for } d > 0 \iff d = k, \text{ and} \\ H_s^k(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \simeq \langle \det_k \rangle \text{ where } \text{res}_E^{\mathfrak{A}_{3k}}(\det_k) = e_1 \wedge \cdots \wedge e_k.$$

and that

$$(2) \quad H_s^*(\mathfrak{A}_{3n+1}, \mathbb{Z}/3\mathbb{Z}) \rightarrow H_s^*(\mathfrak{A}_{3n}, \mathbb{Z}/3\mathbb{Z})$$

is *surjective* (it is injective by (2) of Lemma 3.8).

We prove first the assertions in the displayed formula 1 above, and 2 will follow easily (we just have to check that the determinant class comes from $H_s^*(\mathfrak{A}_{3n+1}, \mathbb{Z}/3\mathbb{Z})$). We apply the Cardenas-Kuhn Theorem 3.3 with $S = \text{Syl}_3(\mathfrak{A}_{3k})$ containing E and $G = \mathfrak{A}_{3k}$. Then

- The fact that $E \simeq (\mathbb{Z}/3\mathbb{Z})^k \subset \text{Syl}_3(\mathfrak{A}_{3k}) \subset \mathfrak{A}_{3k}$ is a closed system has been checked in [Mui], Prop. 2.2: in fact, he checks that if A is any maximal elementary abelian p -subgroup of a symmetric group \mathfrak{S}_n , then any subgroup of a p -Sylow $\text{Syl}_p(\mathfrak{S}_n)$ containing A which is conjugate to A in \mathfrak{S}_n is conjugate to A in $\text{Syl}_p(\mathfrak{S}_n)$. This implies clearly the statement for the alternating groups we need.
- By the Cardenas-Kuhn Theorem or rather its Corollary 3.4, we get that the image of the cohomology of \mathfrak{A}_{3k} in the cohomology of E is

$$\text{im} \left(\text{res}_E^{\text{Syl}_3(\mathfrak{A}_{3k})} : H^*(\text{Syl}_3(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z}) \right) \cap H^*(E, \mathbb{Z}/3\mathbb{Z})^{W_{\mathfrak{A}_{3k}}(E)}.$$

- By Theorem 1.7 and induction

$$\text{res}_E^{\mathfrak{A}_{3k}} : H^*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/3\mathbb{Z})^{W_{\mathfrak{A}_{3k}}(E)}$$

is surjective (compare also the argument in [Mui], Prop. 3.9 and Lemma 3.11).

Thus

$$\text{res}_E^{\mathfrak{A}_{3k}} : H_s^*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \simeq H_s^*(E, \mathbb{Z}/3\mathbb{Z})^{W_{\mathfrak{A}_{3k}}(E)}$$

and by the description of the action of $W_{\mathfrak{A}_{3k}}$ on the stable cohomology of E , we find that only $e_1 \wedge \cdots \wedge e_k$ remains spanning the positive dimensional invariants.

Finally, to prove the surjectivity of the arrow in the displayed formula 2 above, consider the inclusions $E \subset \mathfrak{A}_{3n} \subset \mathfrak{A}_{3n+1}$. Then $\text{Syl}_3(\mathfrak{A}_{3k}) = \text{Syl}_3(\mathfrak{A}_{3k+1})$ and, in exact analogy to the argument above, by [Mui],

Prop. 2.2, $E \subset \mathrm{Syl}_3(\mathfrak{A}_{3k+1}) \subset \mathfrak{A}_{3n+1}$ is a closed system, so that by Cardénas-Kuhn the image of the cohomology of \mathfrak{A}_{3n+1} in the cohomology of E coincides with the image of the cohomology of \mathfrak{A}_{3n} in E (because also $W_{\mathfrak{A}_{3n}}(E) \simeq W_{\mathfrak{A}_{3n+1}}(E)$). \square

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